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Classical backgrounds and scattering for affine Toda theory on a half-line

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Abstract: We find classical solutions to the simply-laced affine Toda equations which satisfy integrable boundary conditions using solitons which are analytically continued from imaginary coupling theories. Both static 'vacuum' configurations and time-dependent perturbations about them which correspond respectively to classical vacua and particle scattering solutions are considered. A large class of classical scattering matrices are calculated and found to satisfy the reflection bootstrap equation.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Integrable Field Theories, Boundary Quantum Field Theory.

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Contents

1. Introduction

Integrable models in two-dimensions possess remarkable features. In particular the S-matrix for such a theory factorises into products of two-particle scattering matrices $S_{cd}^{ab}(\theta)$ (which gives the matrix element for the process $a + b \to c + d$) [28]. Under reasonable physical assumptions, it is possible to show that S must satisfy a number of rather simple algebraic equations. Solutions have been found to these equations for all affine Toda theories and these are postulated to give a non-perturbative expression for the scattering matrix of the theory [5, 10, 7].

These ideas have been extended to include integrable theories on a half-line [6, 13, 16]. In this case it is necessary to introduce another matrix K which describes the reflection of particles off the boundary. Again it is generally believed that all scattering amplitudes factorise into products of S and K. In addition the algebraic equations which were solved by S alone previously can be modified to include K. Recently a number of solutions to these equations for S and K have been found in the context of affine Toda theories [8, 9, 13, 14, 19, 24]. However, many of these solutions make no reference to the boundary conditions which presumably need to be imposed to make sense of the theory. (The exceptions to this are [19] whose perturbative analysis is tied to Neumann boundary conditions and [8, 9]). In fact, integrability places severe constraints on the boundary conditions one can impose; for simply-laced affine Toda theories there are only a finite number of possibilities [8, 9, 3].

Fortunately, there seems to be a straightforward way of analysing which solutions for S and K correspond to which boundary conditions . Whilst S is known to tend to unity in the classical limit, it seems that in general the reflection factor K does not. Thus if we know the classical reflection factors for a particular boundary condition, it should be possible to identify the corresponding quantum reflection factor by considering its classical limit [8].

The aim of this paper is to make some progress towards calculating classical scattering for Toda theories on a half-line. This problem naturally divides into two parts. First it is necessary to find the 'vacuum configuration' or lowest energy solution (which is presumably static) which satisfies a particular boundary condition. Then one solves the linear equations for infinitesimal perturbations about this solution still ensuring that the boundary conditions are satisfied. Far from the wall the solution consists of a superposition of incoming and outgoing waves. The relative phase between the waves is interpreted as the classical limit of the reflection factor K. Some examples of calculations along these lines are to be found in [8] where particular cases within $a_2^{(1)}, d_5^{(1)}$ and $a_1^{(1)}$ were considered.

The paper will be divided into five sections. In the next section we review affine Toda theories, and in particular the boundary conditions which can be imposed which are consistent with the (classical) integrability of the theory [3, 4]. It is then shown that the boundary conditions combine in a particularly neat way with the equations of motion to give linear equations that the tau functions must satisfy at the boundary. This is the key result of the paper which enables us to solve for the classical scattering solutions.

In the third section we consider static solutions to the boundary conditions for the special case of $a_n^{(1)}$ Toda theories. For these theories a large number of tau functions can be constructed explicitly by analytically continuing the multi-soliton solutions of the imaginary coupling theory [18]. Whilst it can be shown that these solutions are singular on the whole line, it is possible that all the singularities can be placed 'behind' the boundary for a theory on the half-line. By substituting this family of tau-functions into the equation derived in the previous section we obtain a large number of possible solutions. The correspondence with particular boundary conditions is discussed.

In section four, whilst remaining within the context of $a_n^{(1)}$ Toda theories, the analysis is extended to cope with the scattering perturbations around vacuum solutions. In this way we are able to derive an expression for the classical reflection matrix. It is shown that this satisfies the classical reflection bootstrap equation.

We conclude with comments on our results and directions for future research.

2. Classical affine Toda theory on a half-line

In this section we briefly review some of the features of affine Toda theory on a half line that we shall need later on to establish notation.

To each affine Kac-Moody algebra $\hat{g}$ we can associate an affine Toda theory [22]. For simplicity in this paper we shall restrict ourselves to simply-laced algebras, and for the most part to $a_n^{(1)}$ which are in some senses the simplest cases of all. The equations of motion for the affine Toda theory associated to $\hat{g}$ are given by

$$\partial_\mu \partial^\mu \phi + \frac{m}{\beta} \sum_{i=0}^{r} n_i \alpha^i e^{\beta \alpha^i \cdot \phi} = 0. \tag{2.1}$$

Here we have used the notation that α^i for $i = 1$ to r are the simple roots of g, the finite Lie algebra associated to $\hat{g}$, and $\alpha^0 = -\psi$, where ψ is the highest root of g (for untwisted algebras). Also we have defined r to be the rank of g and introduced m the mass parameter, and β the coupling constant of the theory. In this paper we shall consider theories for which β is real, and henceforth we set $\beta = m = 1$. The constants n_i are given by the equation

$$\alpha_0 + \sum_{i=1}^{r} n_i \alpha^i = 0 \tag{2.2}$$

and are sometimes referred to as 'marks'.

When the classical theory is considered on the half line $-\infty \leq x \leq 0$, we need to supplement the equations of motion by boundary conditions. In this paper we shall restrict ourselves to conditions of the form

$$\partial_x \phi = F(\phi) \tag{2.3}$$

although more general possibilities can be considered [27, 4]. An arbitrary choice of $F(\phi)$ will generically break the integrability of the system. It turns out that for simply-laced theories (with the exception of sinh-Gordon or $A_1^{(1)}$ affine Toda theory [21, 26]) there are only a finite number of choices for F which preserve integrability [8, 9, 3]. These can be summarised as follows:
(1) Either

$$\partial_x \phi = 0 \ , \tag{2.4}$$

these are 'free' or Neumann boundary conditions.
(2) Or

$$\partial_x \phi = \sum_{i=0}^{r} A_i \alpha^i \sqrt{n_i} e^{\alpha^i \cdot \phi/2} \ , \tag{2.5}$$

where the $A_i = \pm 1$.

The first of these two possibilities is fairly easy to analyse completely at a classical level. The classical energy functional on the half-line is

$$E=\int_{-\infty}^{0}dx\left(\frac{1}{2}(\partial_x\phi)^2+\frac{1}{2}(\partial_t\phi)^2+\sum_{i=0}^{r}n_i(e^{\alpha^i\cdot\phi}-1)\right). \tag{2.6}$$

This is non-negative and the lowest energy configuration is clearly $\phi=0$ which also satisfies the Neumann boundary conditions. If we now take the field $\phi(x,t)=\epsilon(x,t)$ to be some infinitesimal perturbation to this vacuum configuration we see that in a linear approximation we have

$$\partial_\mu\partial^\mu\epsilon+M\epsilon=0\ , \tag{2.7}$$

where $M=\sum_{i=0}^{r}n_i\alpha^i\otimes\alpha^i$ is the mass matrix. The eigenvectors ρ_a, $a=1$ to r, of M are in one-to-one correspondence with the fundamental representations of the Lie algebra g and are interpreted as the basic particle-like excitations of the theory. The eigenvalue of ρ_a is given by $\lambda_a=m_a^2$, where m_a is the mass of the corresponding particle. It is a remarkable fact that the set of masses form the lowest eigenvalue eigenvector of the Cartan matrix of g [5, 11, 12]. In terms of ρ the basic solutions of 2.7 are given by

$$\phi(x,t)=\epsilon(x,t)=\rho_a e^{iEt}(e^{ipx}+K_a e^{-ipx})\ , \tag{2.8}$$

where $E^2-p^2=m_a^2$. This is a superposition of left- and right-moving waves. The classical reflection factor is given by the phase factor K_a which can be determined by substituting the solution (2.8) into the boundary conditions (2.4). This yields the solution $K_a=1$. Classically, at any rate, this boundary condition seems rather uninteresting. There has been considerable progress in understanding the quantum case in [19], where the semi-classical reflection factor has been calculated using perturbative techniques.

In this paper we shall be interested in boundary conditions of the form (2.5). It turns out to be particularly convenient to analyse this case in the language of *tau-functions* [18]. These are introduced as a particular parametrisation of the function ϕ;

$$\phi=-\sum_{i=0}^{r}\alpha^i ln(\tau_i) \tag{2.9}$$

Note that we have introduced $r+1$ functions τ_i to describe the r-component field ϕ and this is reflected in the freedom to scale all the tau-functions $\tau_i\to f(x,t)\tau_i$ without affecting the value of ϕ. This unphysical degree of freedom is fixed by demanding that τ_i satisfies the following particular form of the equations of motion

$$\ddot{\tau}_i\tau_i-\dot{\tau}_i^2-\tau_i''\tau_i+(\tau_i')^2=\left(\prod_{j=0}^{r}\tau_j^{I_{ij}}-\tau_i^2\right)n_i, \tag{2.10}$$

where I_{ij} is the incidence matrix of g given by

$$I_{ij}=2\delta_{ij}-\alpha^i\cdot\alpha^j\ . \tag{2.11}$$

Essentially, I_{ij} takes the value one if the nodes corresponding to i and j are connected on the extended Dynkin diagram, and vanishes otherwise. Substituting the form (2.9) into the boundary conditions (2.5), and taking the inner product with the fundamental weight λ_i, we discover that the boundary conditions can be written in terms of the tau-functions as

$$\frac{\tau_i'}{\tau_i} + \sqrt{n_i} A_i e^{\alpha^i \cdot \phi/2} = n_i C \ , \tag{2.12}$$

where

$$C = (\frac{\tau_0'}{\tau_0} + A_0 e^{\alpha^0 \cdot \phi/2}). \tag{2.13}$$

From this we see that

$$n_i \prod_j \tau_j^{I_{ij}} = \tau_i^2 (n_i C - \frac{\tau_i'}{\tau_i})^2 \tag{2.14}$$

Substituting this into the equations of motion (2.10) we find

$$\ddot{\tau}_i - \frac{(\dot{\tau}_i)^2}{\tau_i} - \tau_i'' + 2n_i C \tau_i' - (n_i^2 C^2 - n_i)\tau_i = 0. \tag{2.15}$$

This equation is the main result of this section. Immediately we see that it has two attractive features. Firstly the equations for τ_i have decoupled so that each equation only involves τ_i for some i. Secondly, if we assume that the solution is static then the first two terms vanish and we are left with the linear equation

$$\tau_i'' - 2n_i C \tau_i' + (n_i^2 C^2 - n_i)\tau_i = 0 \ . \tag{2.16}$$

At this point we perhaps should remind the reader that this equation is deduced from the equations of motion and the boundary conditions, and that the latter are only valid at the boundary $x = 0$. Thus the above equation cannot be solved as a differential equation in x since it is only valid at the boundary, but it will prove a valuable tool in determining the subset of solutions of the equations of motion which satisfy one of the boundary conditions (2.5). One of the drawbacks of the equation (2.16) is that we have essentially squared the boundary conditions, so that we have lost the information contained in the A_i. This we will have to recover by explicitly examining the proposed solutions at the boundary

Perhaps even more surprisingly we can use a similar equation in the case of time-dependent perturbations to a static vacuum solution, i.e. in the situation where

$$\tau_i(x,t) = (\tau_i(x))_{vac} + \epsilon_i(x,t) \ , \tag{2.17}$$

where $\epsilon_i(x,t)$ is some infinitesimal perturbation to the vacuum solution $(\tau_i)_{vac}$. In this case, the important point is that $\dot{\tau}_i = O(\epsilon)$, so discarding terms of $O(\epsilon^2)$ in (2.15) we again arrive at the linear equation

$$\ddot{\tau}_i - \tau_i'' + 2n_i C \tau_i' - (n_i^2 C^2 - n_i)\tau_i = 0 \ . \tag{2.18}$$

We shall use this equation in section four to solve for classical scattering about the vacuum.

One may be concerned that one has introduced a spurious constant C in equation (2.16). Remarkably C has a physical interpretation as proportional to the energy of the solution corresponding to $\tau_i(x,t)$ on the half-line. The energy on the half-line for boundary conditions of type (2.5) is given by

$$E = E_{bulk} - \sum_{i=0}^{r} 2A_i\sqrt{n_i}e^{\alpha^i\cdot\phi/2}|_{x=0} \ , \tag{2.19}$$

where E_{bulk} is the bulk energy given by (2.6) and the second term is the contribution to the energy from the boundary compatible with the boundary conditions (2.5). It has been shown in [23] that for soliton solutions the energy density is a total derivative, so that the bulk energy can be written as a boundary contribution which in terms of tau-functions is

$$E_{bulk} = \left[-2\sum_{i=0}^{r}\frac{\tau_i'}{\tau_i}\right]_{x=-\infty}^{0} . \tag{2.20}$$

In the sections that follow we shall be using tau-functions that tend to a constant as $x\to-\infty$ so the contribution to (2.20) come only from $x=0$. Adding the contribution from the bulk and the boundary we find that

$$E = -2\sum_{i=0}^{r}\left(\frac{\tau_i'}{\tau_i} + A_i\sqrt{n_i}e^{\alpha^i\cdot\phi/2}\right) \tag{2.21}$$

$$= -2\sum_{i=0}^{r} n_i C = -2hC \ , \tag{2.22}$$

where h is the Coxeter number associated to g. Thus, although initially it may have seemed that the appearance of C in the equation (2.16) was a drawback, it turns out that it is an added bonus, giving the energy of the solution that we are considering. This is important in trying to determine the 'vacuum' for a given boundary condition, since the vacuum is defined to be the static solution with the lowest energy which satisfies the boundary condition.

3. Static vacuum configurations for $a_n^{(1)}$ Toda theories from analytically continued solitons

In this section we shall confine ourselves to affine Toda theories based on the algebra $a_n^{(1)}$. We have seen in the previous section that the requirement that a solution satisfies one of the integrable boundary conditions of the form (2.5) can be neatly expressed in terms of tau-functions. For the $a_n^{(1)}$ we have a particularly rich source of such tau-functions which can be constructed by analytically continuing multi-soliton solutions of the imaginary coupling theory [18].

At this point let us briefly review the nature of these solutions in the imaginary coupling theory. Let us reintroduce the coupling constant, so that we write

$$\beta\phi = -\sum_{i=0}^{r} \alpha^i ln(\tau_i) \ . \tag{3.1}$$

The tau-functions for an N-soliton solutions can be compactly written as

$$\tau_j(x,t) = \sum_{\mu_1=0}^{1} \ldots \sum_{\mu_N=0}^{1} \exp\left(\sum_{p=1}^{N} \mu_p \omega^{a_p j} \Phi_p + \sum_{1\le p\le q\le N} \mu_p \mu_q ln A^{(a_p a_q)}\right) . \tag{3.2}$$

Here we have introduced the notation $\Phi_p = \sigma_p(x - v_p t) + \xi_p$ where v_p is the velocity of the p-th soliton, ξ_p is a complex parameter whose real part and imaginary part are related respectively to the position and the topological charge of the p-th soliton. Also σ_p and v_p are related by the mass-shell condition

$$\sigma_p^2(1 - v_p^2) = m_{a_p}^2 \ , \tag{3.3}$$

where a_p labels the species of soliton and $m_{a_p} = 2\sin(\frac{\pi a_p}{n+1})$. We define $\omega = exp(2\pi i/(n+1))$. The variables σ_p and v_p are often conveniently parametrised in terms of the two-dimensional rapidity θ_p by putting

$$\sigma_p = m_{a_p} \cosh(\theta_p) \ , \tag{3.4}$$

$$\sigma_p v_p = m_{a_p} \sinh(\theta_p) \ . \tag{3.5}$$

The constants $A^{(a_p a_q)}$ describe the interaction between the p-th and q-th solitons and are given as

$$A^{(a_p a_q)}(\Theta) = -\frac{(\sigma_p - \sigma_q)^2 - (\sigma_p v_p - \sigma_q v_q)^2 - 4\sin^2\frac{\pi}{n+1}(a_p - a_q)}{(\sigma_p + \sigma_q)^2 - (\sigma_p v_p + \sigma_q v_q)^2 - 4\sin^2\frac{\pi}{n+1}(a_p + a_q)} \tag{3.6}$$

$$= \frac{\sin(\frac{\Theta}{2i} + \frac{\pi(a_p - a_q)}{2(n+1)})\sin(\frac{\Theta}{2i} - \frac{\pi(a_p - a_q)}{2(n+1)})}{\sin(\frac{\Theta}{2i} + \frac{\pi(a_p + a_q)}{2(n+1)})\sin(\frac{\Theta}{2i} - \frac{\pi(a_p + a_q)}{2(n+1)})} \ . \tag{3.7}$$

where $\Theta = \theta_p - \theta_q$. In this section we shall only be interested in static solitons so we shall take $\theta_p = 0$ or equivalently $v_p = 0$, $\sigma_p = m_{a_p}$.

For single solitons the expression (3.2) reduces to

$$\tau_j = 1 + \omega^{ja} e^{\Phi_p} \ . \tag{3.8}$$

The topological charge of such a soliton of species a which is defined by

$$\lim_{x\to\infty}(\phi(x,t) - \phi(-x,t)) \tag{3.9}$$

can be shown to lie in the fundamental representation with highest weight λ_a where $\lambda_a \cdot \alpha^b = \delta_a^b$. It is therefore natural to associate each species of soliton with nodes on the Dynkin diagram of a_n. By analogy with the representation theory of a_n we refer to solitons of type a and type $\bar{a} = n + 1 - a$ as conjugate. We shall (ab)use the notation $\bar{p}$ to denote a soliton whose type is congugate to that of soliton p.

Now let us specialise to the case of interest, namely static solitons. Examining the expression (3.2) carefully we see that solitons obey a kind of Pauli-exclusion principle. We can only construct multi-soliton solutions whose constituent solitons have either different velocity and/or different species. If we attempt to consider two solitons of the same species and velocity, the interaction constant $A^{(aa)}(0)$ vanishes and we simply end up with one constituent soliton of that velocity and species at some different position. This places severe constraints on the possibilities for static soliton configurations; since the velocities of each constituent is identically zero, it follows that each constituent soliton must be of different species. Thus we can consider at most an n-soliton solution where each constituent soliton corresponds to a different node on the Dynkin diagram of a_n. Indeed we believe that this is the most general static solution whose energy-density tends to zero at spatial infinity. The energy of a stationary soliton of type a on the whole line is simply given by

$$E = -\frac{2}{\beta^2}(n+1)m_a \; . \tag{3.10}$$

Note that for imaginary β this is positive, as we might expect. The energy of N stationary solitons is simply given as the sum of energies of the constituent solitons $E = \sum_p E_p$.

The above discussion has all been in the context of imaginary coupling Toda theory, where it is accepted that the field ϕ is allowed to be complex. We are interested in real coupling Toda theory where the field is ϕ is taken to be real. A little thought shows that for tau-functions of the type given in (3.2), reality of ϕ implies the reality of τ_j. This can be ensured by insisting that each constituent soliton p is paired with a congugate soliton $\bar{p}$ with the position/topological charge variables related by $\xi_p = (\xi_{\bar{p}})^*$. An exception to this rule occurs for n odd, where the soliton corresponding to the middle node of the dynkin diagram, i.e. $a_p = (n+1)/2$, is unpaired (since it is its own conjugate), so we must takeξ_p to be real in this case. Thus the reality of the tau-functions closely corresponds to the representation theory of a_n, where only the middle node corresponds to a real representation and the other fundamental representations must be taken in conjugate pairs if we wish to restrict ourselves to real representations. Similar remarks apply equally well to tau-functions associated with other simply-laced algebras

Finally, let us note that the energy E for such solutions on the whole line given in (3.10) is negative for real β. This may seem surprising in view of the manifestly positive energy density of the theory (2.6). This reflects the fact that all such solutions become singular somewhere on the real line, and this is perhaps why they are not generally

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considered in the real coupling theories. However, in the present context the possibility exists that all the singularities in ϕ lie in the 'unphysical' region $x > 0$, so that they are perfectly acceptable physically.

3.1. Two-soliton solutions

As a pedagogical introduction to a more general solution, we begin by considering the possible static two-soliton solutions that satisfy the boundary conditions (2.5). This solution and some of its features have already been discussed in [15]. A two-soliton solution consisting of a soliton of type a and its anti-soliton of type $\bar{a}$ have tau-functions of the form

$$\tau_i = 1 + 2d\cos(\chi + \frac{2\pi i a}{n+1})e^{m_a x} + A^{(a\bar{a})}d^2 e^{2m_a x}\ . \tag{3.11}$$

Here we have set $\chi = Im(\xi)$ and $d = \exp(Re(\xi))$ in the previous notation. Also we calculate $A^{(a\bar{a})}$ from setting Θ to zero in (3.7) as

$$A^{(a\bar{a})} = \cos^2(\frac{\pi a}{n+1}) = 1 - \frac{m_a^2}{4}. \tag{3.12}$$

In general multi-soliton tau-functions (3.2) can be split up into a sum of 'charge' sectors by writing

$$\tau_j = \sum_k T_k \omega^{kj}. \tag{3.13}$$

The linearity of equation (2.16) and the fact that for a_n, we have that $n_i = 1$ for all i implies that each of the T^k separately satisfy the equation;

$$(T_k)'' - 2C(T_k)' + (C^2 - 1)T_k = 0 \quad \text{at} \quad x = 0 \tag{3.14}$$

In the two-soliton case there are three charge sectors:$k = 0, k = \pm a$. Obviously $T_a = \bar{T}_{-a}$, so there are essentially two independent equations;

$$(C^2 - 1) + ((C - 2m_a)^2 - 1)A^{(a\bar{a})}d^2 = 0\ , \tag{3.15}$$

$$((C - m_a)^2 - 1)d = 0\ . \tag{3.16}$$

The second of these equations implies that for a non-trivial solution

$$C_\pm = m_a \pm 1, \tag{3.17}$$

and, substituting this into the first we find that

$$d = \frac{2}{(2 \mp m_a)}. \tag{3.18}$$

The energy of the two solutions are given by $E_\pm = -2hC_\pm = -2(n+1)(m_p \pm 1)$. Actually, it is clear that solutions should come in pairs. The reason is that if $\phi(x,t)$ is a solution to the equations of motion and the boundary conditions, then so is the parity reversed solution $\phi(-x,t)$, since it satisfies boundary conditions of the form (2.5) but with $A_i \to -A_i$. Note that if we sum the energies of the two solutions, the boundary contributions of each solution cancel, and we are left with a bulk contribution of $\phi(x,t)$ and $\phi(-x,t)$ from $-\infty < x < 0$ which can be rewritten as simply the bulk energy of $\phi(x,t)$ on the whole line, and indeed we find that

$$E_+ + E_- = -2(n+1)m_a \, , \tag{3.19}$$

which is the energy of two static solitons of type a.

Before we pick the lower energy solution as the vacuum we should be careful to consider whether the singularities of the corresponding solution for ϕ lie in the physical region $x < 0$ or the unphysical region $x > 0$. Since we have established that the two solutions are parity conjugate, and we know that the solution is singular at some point on the whole line, it follows that at least one solution must be singular somewhere in the physical region. (It is possible that the singularity lies at $x = 0$ which is energetically allowed. We discuss this case later). It turns out that the 'good' solution whose singularities lie in the region $x > 0$ corresponds to the higher energy solution C_-. For this solution the tau-function can be written

$$\tau_j = \frac{(2+m_a) + 4\cos(\chi + \frac{2\pi j a}{n+1})e^{m_a x} + (2-m_a)e^{2m_a x}}{2+m_a} \, . \tag{3.20}$$

Singularities in ϕ occur when τ_j vanishes for some j. Solving the equation $\tau_j = 0$ yields

$$e^{m_a x} = \frac{-\cos(\chi + \frac{2\pi j a}{n+1}) \pm \sqrt{\cos^2(\chi + \frac{2\pi j a}{n+1}) - \cos^2(\frac{\pi a}{n+1})}}{1 - \sin(\frac{\pi a}{n+1})} \, . \tag{3.21}$$

For a real solution for x this implies that

$$\cos^2(\chi + \frac{2\pi j a}{n+1}) \geq \cos^2(\frac{\pi a}{n+1}) \, , \tag{3.22}$$

$$\cos(\chi + \frac{2\pi j a}{n+1}) \leq 0 \, . \tag{3.23}$$

Combining these two conditions gives

$$-1 \leq \cos(\chi + \frac{2\pi j a}{n+1}) \leq -\cos(\frac{\pi a}{n+1}) \, . \tag{3.24}$$

It is easy to check that for any choice of χ there will be some j for which this condition is satisfied, so that there will always be a singularity at some real value of x for some j. To see where these singularities lie consider the right hand side of (3.21) where the square-root is taken with the minus sign, since this is the smaller of the two solutions, and hence the more likely to yield negative values of x. A simple calculation shows

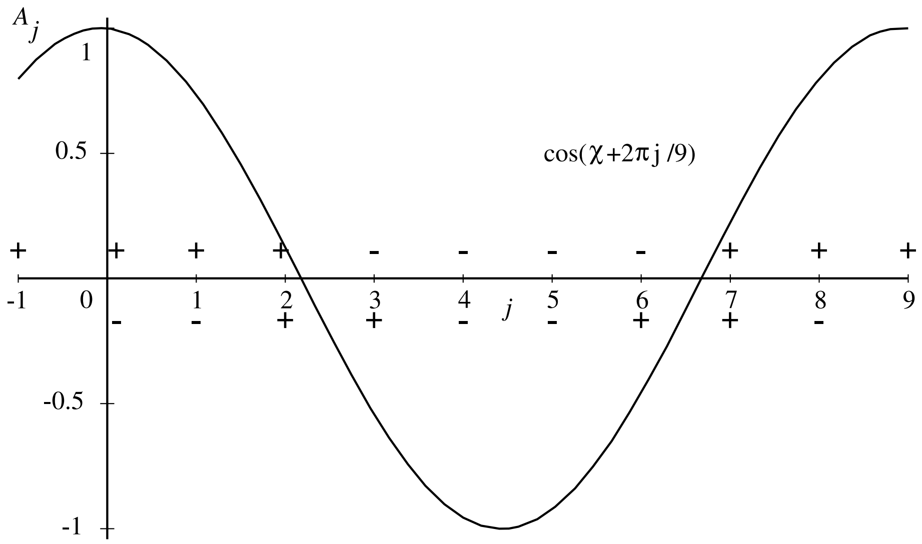

Figure 1: Boundary conditions for $N = 1$, $n = 8$ and $a = 2$

that as a function of $\cos(\chi + \frac{2\pi j a}{n+1})$ the right hand side is a monotonically increasing function, so its lowest value is attained when $\cos(\chi+\frac{2\pi j a}{n+1}) = -1$. At this point the right hand side is equal to one, corresponding to a singularity at $x = 0$. For other values of $\cos(\chi+\frac{2\pi j a}{n+1})$, the right hand side will be greater than one, and the singularities will be in the unphysical region $x > 0$.

Having found a class of two soliton solutions which correspond to non-singular fields ϕ, it only remains to ascertain precisely which boundary conditions each solution satisfies. Substituting $x = 0$ into the expression (3.20), we find that the tau-function at this point can be written as

$$\tau_j = \frac{\cos^2(\frac{\chi}{2} + \frac{\pi j a}{n+1})}{1 + \sin(\frac{\pi a}{n+1})} . \tag{3.25}$$

Thus we can write

$$e^{\alpha^j \cdot \phi} = \frac{\cos^2(\frac{\chi}{2} + \frac{\pi(j+1)a}{n+1}) \cos^2(\frac{\chi}{2} + \frac{\pi(j-1)a}{n+1})}{\cos^4(\frac{\chi}{2} + \frac{\pi j a}{n+1})} . \tag{3.26}$$

In the boundary conditions (2.5), we have terms involving $\exp(\alpha^j \cdot \phi/2)$ which should be interpreted as the positive square root of the above quantity. The values of A_i are found to be given by

$$A_j = -\mathrm{sign}\left(\cos(\frac{\chi}{2} + \frac{\pi(j+1)a)}{n+1}) \cos(\frac{\chi}{2} + \frac{\pi(j-1)a)}{n+1})\right) . \tag{3.27}$$

In figure 1 we have plotted $\cos(\frac{\chi}{2} + \frac{\pi j a}{n+1})$, tabulated its sign, and tabulated the sign of the right hand side of (3.27). From this we can see that the A_j is generically negative except at near a point where $\cos(\frac{\chi}{2}+\frac{\pi j a}{n+1})$ has a zero, where a pair of positive A_j are produced. The number of positive pairs is related to the number of such zeroes which are in turn related to the value of a. One also observes that the expression (3.27) is invariant under

$$\chi \to \chi - \frac{\pi a}{n+1} \tag{3.28}$$

$$j \to j+1 \ . \tag{3.29}$$

This implements the Z_n symmetry of $A^{(1)}$ Toda theories, and cyclically permutes the boundary conditions $A_j \to A_{j+1 \bmod n+1}$.

3.2. Multi-soliton solutions

In this section we shall generalise the two-soliton solution given above to multi-soliton solutions with arbitrarily many constituent solitons. Whilst this case is much more complicated, and we cannot guarantee finding a complete set of solutions to the basic equation (2.16), we shall give a wide class of explicit solutions.

To begin with we shall restrict our attention to multi-soliton solutions with an even number of constituent solitons; that is, we shall assume that the soliton corresponding to the middle node of the Dynkin diagram for n odd does not feature. Thus our ansatz consists of N pairs of conjugate solitons which we shall label p and $\bar{p}$ respectively. The real tau-function can be written in the form

$$1+2\sum_{p=1}^{N} d_p \cos(\chi_p+\frac{2\pi a_p}{n+1})e^{m_{a_p}x}+\dots \ , \tag{3.30}$$

where the dots compactly represent the (many) interaction terms. As in the two-soliton case we can decompose the tau-function into sectors as in (3.13). As a further simplification we shall assume that n is large with respect to the charges of the solitons, or more particularly that the largest charge $Q_{max}=\sum_p a_p \le (n+1)/2$. With this restriction, the charges in (3.13) take values between Q_{max} and $-Q_{max}$, and only the term

$$\prod_{p=1}^{N}\left(d_p e^{i(\chi_p+\frac{2\pi a_p}{n+1})}\right)\left(\prod_{1\le p<q\le N} A^{(pq)}\right)\exp\left(\sum_{p=0}^{N} m_{a_p}x\right) \tag{3.31}$$

contributes to the charge Q_{max}. Substituting this into the equation (3.14), we immediately find

$$C_{\pm}=\sum_{p=0}^{N} m_{a_p} \pm 1 \tag{3.32}$$

and the energy is as usual $-2(n+1)C_{\pm}$. Once more this pair of solutions are the parity inverses of one another. To determine the 'positions' of the constituent solitons we consider the terms with charge $Q_p = Q_{max}-a_p$. Again we make the assumption the charges are chosen in such a way that only two terms in the τ_j contribute to T_{Q_p}: namely the term containing all the solitons except the p-th, and the term containing all the solitons and the p-th conjugate soliton. More explicitly we have

$$T_{Q_p} = \prod_{r\neq p}\left(D_r e^{i(\chi_r+\frac{2\pi a_r}{n+1})}\right)\left(\prod_{r<s:r,s\neq p} A^{(a_r a_s)}\right)\exp\left(\sum_{r\neq p} m_{a_r} x\right)\times$$
$$\times\left(1 + d_p^2 A^{p\bar{p}}\prod_{r\neq p}\left(A^{(rp)}A^{(r\bar{p})}\right)e^{2m_{a_p}x}\right) . \tag{3.33}$$

Substituting this into (3.14) we discover that

$$(d_p)^2 A^{p\bar{p}}\prod_{r\neq p}\left(A^{(rp)}A^{(r\bar{p})}\right) = \frac{2\pm m_{a_p}}{2\mp m_{a_p}} . \tag{3.34}$$

One can prove that

$$A^{(rp)}A^{(r\bar{p})} = \frac{(m_{a_r}-m_{a_p})^2}{(m_{a_r}+m_{a_p})^2} , \tag{3.35}$$

so that

$$d_p = \prod_{r\neq p}\frac{(m_{a_r}+m_{a_p})}{|m_{a_r}-m_{a_p}|}\frac{2}{2\mp m_{a_p}} . \tag{3.36}$$

Here we have made a choice for the sign of d_p, since we can absorb this sign into an $i\pi$ shift in χ_p.

Strictly speaking we have only given necessary conditions that some of the charge sectors in the tau-function satisfy (3.14), but extensive numerical investigation supports the conjecture that (3.32) and (3.36) provide a solution to (2.16). Moreover whilst we enforced fairly stringent conditions on a_p to derive the necessary constraints on d_p and C, many examples seem to confirm the idea that these values give a solution even when the conditions are not met. However, in those cases one generally might expect other solutions not of the form (3.32), (3.36).

By analogy with the two-soliton case, we should proceed by discussing the position of the singularites in ϕ. Since the solutions corresponding to $C_\pm$ are parity conjugate, and we know from the negative energy that ϕ is singular somewhere on the real line $-\infty < x < \infty$, we know that one of the two solutions must be singular in the physical region $x < 0$. Unfortunately, finding the zeroes of the tau-function in general is very difficult. Numerical work suggests that it is more difficult to find regular solutions as more solitons are introduced. As alluded to above, solitons in the real coupling theory have negative energy, so that it seems that the lowest energy configuration consists of adding in as many solitons as possible without making the solution singular for $x < 0$. One possibility is that the best we can do is to place the singularity at the wall. In this case the analysis leading to the expression for the total energy (2.22) is still correct and yields a finite answer. Physically the infinities in the bulk and boundary energies exactly cancel.

In parallel with the two-soliton case, one can ask which boundary condition does the solution given above correspond to; that is for what A_i does the above solution satisfy the boundary condition. Again the answer seems to be a lot more complicated than for two-solitons, but nonetheless some of the features of that case do seem to persist. For the two-soliton case it was seen that one could write $\tau_j(x=0)$ in the form LW_j^2 where all the j-dependence is in W. From this we deduced that

$$e^{\alpha^j\cdot\phi/2}|_{x=0} = \frac{|W_{j-1}W_{j+1}|}{W_j^2} \tag{3.37}$$

and that

$$A_j = \pm\mathrm{sign}(W_{j-1}W_{j+1}) \ , \tag{3.38}$$

where the plus/minus sign corresponds to the two solutions $C_\pm$. We postulate that we can write the tau-function (3.30) evaluated at $x=0$ in the form LW_j^2 where we can expand W as

$$W_j = \sum_{\sigma_1=1,\sigma_p=\pm1} c_{\sigma_1\sigma_2\ldots\sigma_N}\cos\Big(\sum_p \sigma_p\Big\{\frac{\chi_p}{2}+\frac{\pi a_p j}{n+1}\Big\}\Big) \ . \tag{3.39}$$

By comparing this with the tau-function with d_p given in (3.36), we find

$$c_{\sigma_1\sigma_2\ldots\sigma_N} = \prod_{r<s}\sin\Big(\frac{\pi}{n+1}(|\sigma_r a_r - \sigma_s a_s|)\Big) \tag{3.40}$$

and

$$L = 4\prod_r \frac{2}{(2\mp m_{a_r})}\prod_{r<s}\frac{4}{(m_{a_r}-m_{a_s})^2}. \tag{3.41}$$

We have checked that this ansatz for $\tau_j(0)$ is true up to six solitons. By combining (3.40),(3.39),(3.38) we can deduce A_j for any given values of χ_p, and use this to deduce a 'phase' diagram for different boundary conditions as a function of the χ_p. One such example is given in figure 2, where we have taken $n=8, N=2, a_1=1, a_2=3$. In this figure we have plotted the lines in the parameter space (χ_1,χ_2) along which $\tau_j(0)=0$ for $j=0,..,8$.

In the region marked I the boundary conditions are found to be

$$(A_0, A_1, .., A_8) = (-1,1,1,-1,-1,-1,-1,1,1). \tag{3.42}$$

The boundary conditions for the other regions can be deduced by noting that W_j changes sign as the line $\tau_j=0$ is crossed. So by (3.38), A_{j-1} and A_{j+1} change sign when this line is crossed. One can check that the resulting boundary conditions are compatible with the Z_n symmetry which is realised as

$$\chi_p \to \chi_p - \frac{2\pi a_p}{(n+1)} \ , \tag{3.43}$$

$$j \to j+1 \ . \tag{3.44}$$

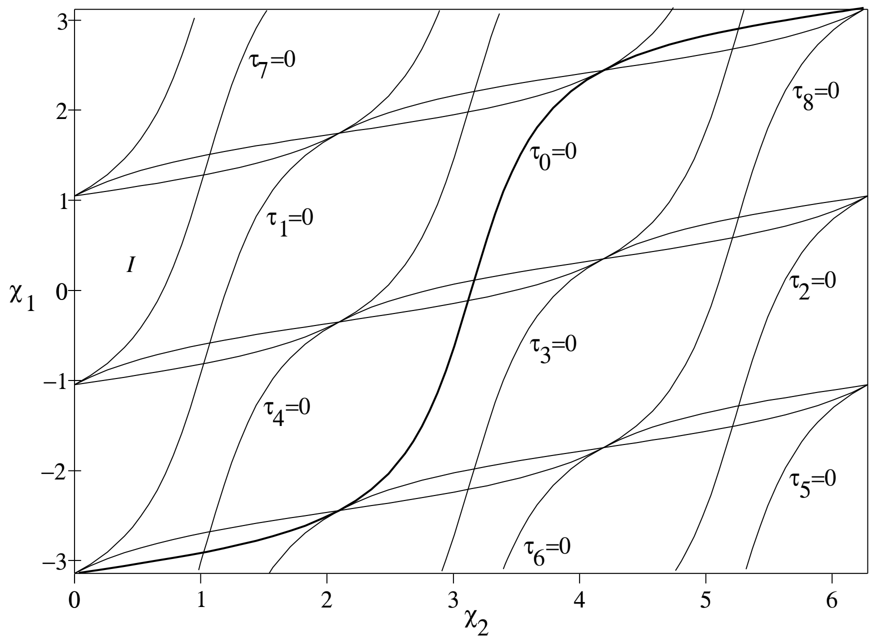

Figure 2: Boundary conditions for $N = 2$, $n = 8$, $a_1 = 1$, $a_2 = 3$

Suppose that we choose χ_p so that we lie on one of the lines $\tau_j = 0$. Then the corresponding solution for ϕ is singular, and in fact

$$e^{\alpha^j \cdot \phi} = +\infty, \tag{3.45}$$

$$e^{\alpha^{j+1} \cdot \phi} = 0, \tag{3.46}$$

$$e^{\alpha^{j-1} \cdot \phi} = 0. \tag{3.47}$$

Thus along this line the coefficients A_{j-1} and A_{j+1} are undetermined as we may have expected. It follows that a particularly rich variety of boundary conditions are allowed at the points where many such lines intersect.

If n is odd we can consider static soliton solutions containing a single constituent soliton of type $a_{N+1} = (n+1)/2$. Reality of the tau-function constrains the associated parameter ξ_{N+1} to be real, or equivalently $\chi_p = 0, \pi$. Following much the same argument as in the previous section we initially assume that the $a_p = n + 1 - a_{\bar{p}}$, $p = 1, .., N$ are chosen in such a way that only one term contributes to the sector of maximal charge $Q_{max} = \sum_{p=1}^{N} a_p + a_{N+1}$ and two terms contribute to sectors of charge $Q_p = Q_{max} - a_p$, $p = 1, .., N$. This results in similar expressions for $C_\pm$ and d_p, $p = 1, .., N$ except that the sums in (3.32), (3.36) run from $1, .., N + 1$. It only remains to determine the value of d_{N+1}. Only one term contributes to the charge Q_{N+1} sector since as there is no soliton conjugate to the middle soliton $p = N + 1$, we cannot add a soliton-conjugate soliton pair to the term involving solitons of type $p = 1, .., N$. Instead the resulting equation is

$$0 = (C_\pm - \sum_{p=1}^{N} m_{a_p})^2 - 1 = (m_{a_{N+1}} \pm 1)^2 - 1\ . \tag{3.48}$$

Since $m_{a_{N+1}} = 2\sin(\pi/2) = 2$, we see that only the C_- solution is allowed. The value of d_{N+1} is unconstrained by this or any other equation, so we are free to place the middle soliton anywhere! Under a parity transformation, the energy and the values of d_p, $p = 1, .., N$ remain unchanged. This is in agreement with the idea that the energy of the solution on the whole line is formally given by the sum of the half-line energies of the solution and its parity conjugate; that is in this case

$$-2(n+1)(C_- + C_-) = -2(n+1)(2\sum_{p=1}^{N+1} m_{a_p} - 2) = -2(n+1)(2\sum_{p=1}^{N} m_{a_p} + m_{a_{N+1}}) \quad (3.49)$$

as we should have expected from the masses of the constituent solitons.

4. Scattering solutions for $a_n^{(1)}$ Toda theories

In the previous section we showed how one can construct static background configurations. In this section we show how to solve the classical linearised perturbation equations around this background, and explicitly calculate the classical scattering matrix. Once more we shall rely on the tau-functions given by multi-soliton solutions.

Let us assume that the field is infinitesimally perturbed about the vacuum configuration

$$\phi(x,t) = \phi(x)_{vac} + \epsilon(x,t). \quad (4.1)$$

Substituting this into the equations of motion and the boundary conditions (2.5), and keeping terms linear in $\epsilon(x,t)$ yields

$$\partial_\mu\partial^\mu\epsilon(x,t) + \sum_{i=0}^{r} n_i\alpha^i e^{\alpha^i\cdot\phi_{vac(x)}}(\alpha^i\cdot\epsilon(x,t)) = 0 \quad (4.2)$$

and

$$\partial_x\epsilon(x,t) = \frac{1}{2}\sum_{i=0}^{r} A_i\alpha^i\sqrt{n_i}e^{\alpha^i\cdot\phi_{vac}(x)/2}(\alpha^i\cdot\epsilon(x,t))\ . \quad (4.3)$$

Far away from the boundary $x = 0$, the field $\phi_{vac}(x,t) \to 0$ for finite energy configurations, and in this limit the equation (4.2) reduces to (2.7) and the solution for $\epsilon(x,t)$ tends to

$$\epsilon(x,t) \to \rho_a e^{-iEt}(e^{ipx} + K_a e^{-ipx})\ , \quad (4.4)$$

where $E^2 - p^2 = m_a^2$. This solution consists of the superposition of two oscillatory solutions corresponding to incoming and outgoing 'waves' or quantum mechanically 'particles'. On the other hand, the field $\phi(x,t)$ corresponding to a single soliton of type a has the asymptotic form

$$\phi(x,t) \sim \rho_a e^{\Phi(x,t)}\ , \quad (4.5)$$

where $\Phi(x,t)=\sigma(x-vt)$, and $\sigma^2-\sigma^2v^2=m_a^2$. Comparison of the two asymptotic behaviours suggests that we can obtain appropriate oscillatory solutions from the soliton tau-functions if we make the identification

$$\sigma = \pm ip \tag{4.6}$$

$$\sigma v = iE. \tag{4.7}$$

From here it is clear how to proceed. We take the tau-function τ_{vac} corresponding to the static vacuum solution that we found in the previous section and add in two further time-dependent solitons with the identifications (4.7). If we label the incoming and outgoing solitons by indices I, and O respectively, then we take

$$\sigma_I = -\sigma_O = ip \ \ , \ \ \sigma_I v_I = \sigma_O v_O = iE \ , \tag{4.8}$$

$$d_I = d_O = \epsilon \ \ , \ \ \chi_I = -\chi_O = \psi \ \ , \ \ a_I = a_O = b \tag{4.9}$$

Note that these two solitons are *not* a conjugate pair but are both of species b. The relative phase between the two waves is given by

$$K^b = e^{-2i\psi} \ . \tag{4.10}$$

This tau-function describes the scattering of a particle of species b on the background given by τ_{vac}. As described in section two, requiring that this tau-function satisfies one of the boundary conditions (2.5) up to $O(\epsilon)$ amounts to the equation

$$\ddot{\tau}_i - \tau_i'' + 2C\tau_i' - (C^2-1)\tau_i|_{x=0} = 0. \tag{4.11}$$

Considering the cases where the vacuum solution has even/odd number of constituent solitons in turn, let us assume first that it has $2N$ solitons. The linearity of equation (4.11 implies that we can divide τ_j into charge sectors as in the previous equation, and use the equation (3.14) with the addition of a time-derivative term as in(4.11), i.e.

$$\ddot{T}_k - T_k'' + 2CT_k' - (C^2-1)T_k|_{x=0} = 0. \tag{4.12}$$

There are at least two highest charge terms in the tau-function with charge $Q_{max} = \sum_{p=1}^N a_p + b$ of the form

$$\begin{aligned} T_{Q_{max}} = {} & \prod_{j=1}^N d_p \prod_{1\le j<k\le N} A^{(jk)} e^{i\sum_{j=1}^N \chi_j} e^{(\sum_{j=1}^N m_{a_j})x} \\ & \times \left(e^{i\psi} e^{iEt+ipx} \prod_{j=1}^N A^{(a_j b)}(p) + e^{-i\psi} e^{iEt-ipx} \prod_{j=1}^N A^{(a_j b)}(-p) \right) \end{aligned} \tag{4.13}$$

where $A^{(a_j b)}(p)$ are calculated using the definition (3.7), (4.7). Substituting this expression into (4.12) we find that the scattering matrices on the two solutions corresponding to $C_\pm$ are given by

$$K_\pm^b = e^{-2i\psi} = \frac{2ip \mp m_b^2}{2ip \pm m_b^2} \frac{\prod_{j=1}^N A^{(a_j b)}(p)}{\prod_{j=1}^N A^{(a_j b)}(-p)}. \tag{4.14}$$

If the vacuum solution has an odd number of solitons, then we find that the scattering matrix is given essentially by the above expression for K_-, but where the product runs from $1,..,N+1$. Note that the reflection factor given by (4.14) only depends on the number and species of solitons in the background solution, not on the topological charge parameters χ_p. It follows that boundary conditions related by the Z_n symmetry (3.44) will have identical scattering matrices.

4.1. Classical reflection bootstrap equations

In the introduction, it was pointed out that in two dimensions, integrability placed strong constraints on the S-matrix of the theory, and that S and K satisfied various algebraic constraints. In the case at hand both S and K are diagonal and in the classical limit S tends to unity, and these two facts ensure that most of the algebraic constraints are satisfied automatically. However, one non-trivial check is the reflection bootstrap equation, which in the classical limit becomes

$$K^c(\theta_c) = K^a(\theta_c + i\bar{\theta}^b_{ac})K^b(\theta_c - i\bar{\theta}^a_{bc}) \;, \tag{4.15}$$

where the fusion angles are given by

$$\theta^b_{ac} = \frac{\pi(a+b)}{2(n+1)} \;, \tag{4.16}$$

$$\bar{\theta} = \pi/2 - \theta \tag{4.17}$$

and $a+b+c = n+1$. (For simplicity we ignore the case that $a+b+c = 2(n+1)$. For more details about fusion angles see for instance [5] and about the classical reflection equation see [13, 8].)

In the equation (4.15), the argument of K^c is the usual rapidity variable ϕ which is related to the momentum p and energy E of the incoming particle of type a by the formulae

$$p = m_a \sinh(\phi) \;, \tag{4.18}$$

$$E = m_a \cosh(\phi). \tag{4.19}$$

Note that using the identification (4.7), we see that $\theta_p = \phi + i\frac{\pi}{2}$ where θ_p appears in(3.5). If, as is usual, one defines a bracket notation

$$(x) = \frac{\sinh(\frac{\phi}{2} + \frac{xi\pi}{2(n+1)})}{\sinh(\frac{\phi}{2} - \frac{xi\pi}{2(n+1)})} \;, \tag{4.20}$$

then one can write in this notation

$$\frac{A^{(a_j b)}(p)}{A^{(a_j b)}(-p)} = \frac{(\frac{n+1}{2} + a_j - b)(\frac{n+1}{2} - a_j + b)}{(\frac{n+1}{2} + a_j + b)(\frac{n+1}{2} - a_j - b)} \;, \tag{4.21}$$

where the left hand side of this equation is one of the factors appearing in the scattering matrix $K^b_{\pm}$ in equation (4.14). A straightforward calculation shows that each of these factors individually satisfy the reflection bootstrap equation. To prove that K^b satisfies (4.15) it remains only to show that the remaining factor

$$\frac{2ip - m_b^2}{2ip + m_b^2} = -\frac{1}{(n+1-a_b)(a_b)} \tag{4.22}$$

satisfies the bootstrap equation, and this is indeed the case.

5. Other simply-laced algebras

In this section we shall make a few remarks about extending the results found in the previous two sections to affine Toda theories based on the D and E series of algebras. The major technical difficulty in extending the results to these algebras is that there is no explicit form for tau-functions which correspond to more than two-soliton solutions, and even those solutions that are known are considerably more complicated than those of the $a_n^{(1)}$ theories [17]. Because of this, we shall restrict our search to finding which single soliton solutions [2] can be found which satisfy the boundary conditions (2.5). The general form for the corresponding tau-functions is

$$\tau_j = 1 + \delta_1^{(p)} d_p e^{m_p x} + ... + \delta_{n_j}^{(p)} (d_p)^{n_j} e^{n_j m_p x} \ , \tag{5.1}$$

where p labels the species of soliton as before. The coefficients $\delta_i^{(p)}$ have been explicitly calculated for all affine algebras on a case by case basis, and they are found to be real if and only if the minimal representation λ_p of the corresponding finite Lie algebra is real. Restricting ourselves to such real tau-functions, we substitute the tau-functions (5.1) case by case into the equation (2.16) and solve for C and d_p. The results are presented below. The labelling of nodes on the Dynkin diagram is given in figure 3.

$d_4^{(1)}$

The only solution is associated with the triality invariant soliton of species $p = 2$. The corresponding tau-functions are given by

$$\tau_0 = \tau_1 = \tau_3 = \tau_4 = 1 + d_2 e^{\sqrt{6}x}, \ \text{ and } \ \tau_2 = 1 - 4d_2 e^{\sqrt{6}x} + (d_2)^2 e^{2\sqrt{6}x} \tag{5.2}$$

and we find the solutions $C = \sqrt{6}/2, \sqrt{6}/2 \pm 3\sqrt{2}/2, \sqrt{6}/2 \pm \sqrt{2}/2$ and correspondingly $d_2 = -1, (5 \pm 3\sqrt{3})/(-5 \pm 3\sqrt{3}), 2 \pm \sqrt{3}$. Only the solutions corresponding to $-1, (5 - 3\sqrt{3})/(-5 + 3\sqrt{3}), 2 - \sqrt{3}$ are singularity free in the region $x < 0$.

For $d_2 = -1, \tau_0 \to 0$ as $x \to 0$, so that the only non vanishing component of $\lambda_2 \cdot \phi \to -\infty$ in this limit. Nonetheless the solution has finite energy, and obeys the

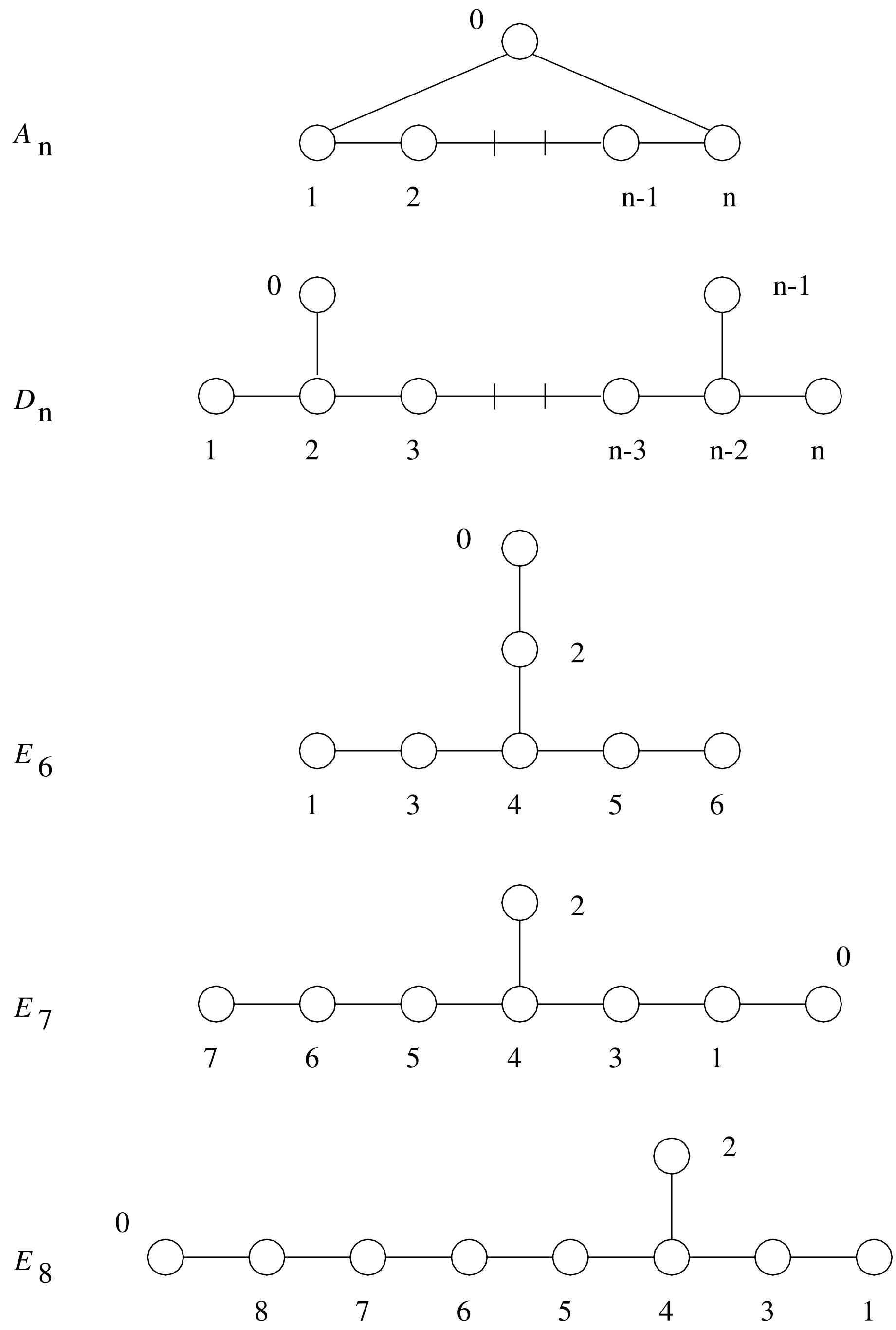

Figure 3: Root labels for simply-laced algebras

boundary condition $A_0 = A_1 = A_3 = A_4 = -1$ and A_2 is unspecified since $e^{\alpha_2 \cdot \phi}$ vanishes at the wall. For $d_2 = 2 - \sqrt{3}$ we find that τ_2 vanishes at the wall so that $\lambda_2 \cdot \phi \to \infty$. We find that A_0, A_1, A_3, A_4 are unconstrained and $A_2 = 1$. Finally the solution with $d_2 = (5 - 3\sqrt{3})/(-5 + 3\sqrt{3})$ is regular at the wall, and obeys boundary conditions with $A_i = -1$ for all i.

$d_5^{(1)}$

Again the only solution is associated with the species $p = 2$. The corresponding tau-functions are given by

$$\tau_0 = \tau_1 = \tau_4 = \tau_5 = 1 + d_2 e^{2x}, \quad \text{and} \quad \tau_2 = \tau_3 = 1 - 2d_2 e^{2x} + (d_2)^2 e^{4x}. \tag{5.3}$$

We find the solutions $C = (1 \pm \sqrt{2 \pm \sqrt{2}})$. The corresponding values of d_2 are $(2 \pm \sqrt{2 \pm \sqrt{2}})/(2 \mp \sqrt{2 \pm \sqrt{2}})$. The solutions which are regular in the region $x < 0$ are those with $C = (1 - \sqrt{2 \pm \sqrt{2}})$. These solutions satisfies the boundary condition with $A_0 = A_1 = A_4 = A_5 = -1$ and $A_2 = A_3 = \mp 1$.

$d_n^{(1)}$**,** $n > 5$

The solitons corresponding to the tip nodes do not yield any solutions. The other solitons correspond to roots α_p with $n_p = 2$ and their tau-functions are given by the formulae

$$\tau_0 = -\tau_1 = 1 + d_p e^{m_p x}\ , \tag{5.4}$$

$$\tau_i = 1 + 2\frac{\cos(\frac{(2i-1)\pi p}{2(n-1)})}{\cos(\frac{\pi p}{2(n-1)})} d_p e^{m_p x} + (d_p)^2 e^{2m_p x} \quad , 2 \leq i \leq n-2\ , \tag{5.5}$$

$$\tau_{n-1} = \tau_n = 1 + (-1)^p d_p e^{m_p x}\ . \tag{5.6}$$

The value of the coefficient involving cosine takes at least two different values for different i, so inserting the difference of two such distinct tau-functions into (2.16) immediately yields that $2C = m_p \pm \sqrt{2}$. Now substituting τ_0 and τ_2 into the same equation we find that $m_p = \sqrt{6}$ and that $d_p = 2 \pm \sqrt{3}$. Given that

$$m_p = 2\sqrt{2}\sin(\frac{\pi p}{2(n-1)})\ , \tag{5.7}$$

we find that we must have $n = 3x + 1$, $p = 2x$. With these values (5.6) reduces to

$$\tau_0 = -\tau_1 = \tau_{n-1} = \tau_n = 1 + d_p e^{\sqrt{6}x}\ , \tag{5.8}$$

$$\tau_{3i} = \tau_{3i+1} = 1 + 2d_p e^{\sqrt{6}x} + (d_p)^2 e^{2\sqrt{6}x}\ , \tag{5.9}$$

$$\tau_{3i+2} = 1 - 4d_p e^{\sqrt{6}x} + (d_p)^2 e^{2\sqrt{6}x}\ . \tag{5.10}$$

Only the solution with $d_p = 2 - \sqrt{3}$ is non-singular in the region $x < 0$. The solution is compatible with the boundary conditions $A_{3i+2} = -1$, and the other A_i are unconstrained.

$e_6^{(1)}$

In this case only the solitons of species $p = 2, 4$ have real tau-functions. Solutions which satisfy the boundary conditions can be found in both cases. If $p = 2$, then $m_2 = \sqrt{6 + 2\sqrt{3}}$ and solutions can be found with $C = m_2/2, m_2/2 \pm \sqrt{6 - 2\sqrt{3}}/2$ and $d_2 = -1, 5 \pm 2\sqrt{6}$. Only the solution with $d_2 = 5 - 2\sqrt{6}$ is non-singular for $x < 0$, and satisfies boundary conditions (2.5) with $A_4 = -1$ and the other $A_i = 1$.

If $p = 4$, then $m_2 = \sqrt{6 - 2\sqrt{3}}$ and solutions can be found with $C = m_2/2, m_2/2 \pm \sqrt{6 + 2\sqrt{3}}/2$ and $d_4 = -1, 5 \pm 2\sqrt{6}$. The solutions with $d_4 = -1, 5 - \sqrt{6}$ are regular for $x < 0$ and satisfy the boundary conditions $A_0 = A_1 = A_6 = A_4 = 1$ and the other A's unspecified, and $A_i = 1$ for all i respectively.

$e_7^{(1)}$

Only solitons of species $p = 3$ and $m_3 = \sqrt{6}$ can satisfy the boundary conditions with $C = m_3/2, m_3 \pm \sqrt{2}/2$. The corresponding values of d_3 are $-1, 2 \pm \sqrt{3}$ and only the solutions with $d_3 = -1, 2 - \sqrt{3}$ are free of singularities for $x < 0$. The solutions with $d_3 = -1$ does not seem to correspond to any sensible boundary condition of the type (2.5) but for $d_3 = 2 - \sqrt{3}$ we find that $A_1 = A_2 = A_6 = -1$ with the other A_i unconstrained.

$e_8^{(1)}$

By using the two tau-functions corresponding to $n_i = 2$ and also τ_0, one can use similar arguments to those used for $d_n^{(1)}$, $n > 5$ to deduce that $m_p = \sqrt{(6)}$. Since there are no single solitons with such a 'mass' we find a contradiction so that there are no single-soliton solutions.

5.1. Scattering

In section four we showed how by adding in two further solitons with imaginary momentum, we could describe classical scattering solutions on the static 'vacuum' configurations. The tau-functions had to obey the equation

$$\ddot{\tau}_i - \tau_i'' + 2Cn_i\tau_i' - (C^2n_i^2 - n_i)\tau_i|_{x=0} = 0 \ . \tag{5.11}$$

As explained above, multi-soliton tau-functions for algebras other than $a_n^{(1)}$ are very complicated in general. However, the exception to this rule is τ_0, which has exactly the same form as for the $a_n^{(1)}$ case, except that the interaction coefficient $A^{(pq)}(\theta)$ defined in (3.7) must be generalised. For a definition of the generalisation $X^{(pq)}(\theta)$ and a discussion of the properties it enjoys please see reference [20]. Restricting ourselves to single soliton backgrounds, and to linear order in the perturbation parameter ϵ we find that

$$\begin{aligned}\tau_0 = 1 + d_r e^{m_r x} + \epsilon e^{i\psi} e^{-iEt+ipx} + \epsilon e^{-i\psi} e^{-iEt-ipx} + d_r \epsilon e^{i\psi} e^{-iEt+ipx+m_r x} X^{(rb)}(p) + \\ + d_r \epsilon e^{-i\psi} e^{-iEt-ipx+m_r x} X^{(rb)}(-p) \ . \end{aligned} \tag{5.12}$$

Substituting this into (5.11) and looking at the term linear in ϵ one recovers

$$e^{-2i\psi} = \frac{\big((C-m_r)^2-1)((C-m_r)^2+2ip(C-mr)+m_b^2-1)X^{(rb)}(p)-(C^2-1)(C^2+2ipC+m_b^2-1)\big)}{\big((C-m_r)^2-1)((C-m_r)^2+2ip(C-mr)+m_b^2-1)X^{(rb)}(p)-(C^2-1)(C^2+2ipC+m_b^2-1)\big)} \ . \tag{5.13}$$

6. Conclusions

We have seen that integrable boundary conditions (2.5) and the equations of motion combine neatly to yield a simple equation satisfied by the tau-functions of affine Toda theory. In the case of $A_n^{(1)}$ a large class of static solutions were found, and moreover it proved relatively straightforward to extract the scattering data on these backgrounds. As a consistency check we showed that the scattering matrices satisfy the classical reflection boostrap equation.

Still, many open questions remain even in this simple case. We only provided a plausible argument that the conjectured form of tau-function did indeed satisfy the boundary condition, and we did not rule out the possibility of other solutions, amongst which the true vacuum solution may lie . The solutions consisting of $2N$ solitons contained N unspecified parameters χ_p (those with $2N+1$ solitons were specified by $N+1$ parameters). These can be thought of as moduli for zero-modes, since the energy of the solutions is independent of χ_p. We can change the boundary conditions satisfied by the solutions by varying these parameters, in essentially the same way that one varies the topological charge of solitons in the imaginary coupling theory. The 'phase' diagram which specifies which boundary conditions can be obtained from a particular family of solutions by varying χ_p seems in general very complicated and this makes it difficult to isolate the true vacuum solution. Nonetheless, the scattering data is found to be independent of the topological charge, so all the resulting boundary conditions will share the same scattering matrix K.

A related difficulty is to find which solutions are singularity free in the region $x < 0$. The general rule seems to be that as we add in more solitons the energy of the solution decreases rather than increases because of the reality of the coupling, but also the solution develops more singularities. The trick to finding the true vacuum is therefore to add in as many solitons as one can without introducing singularities into the physical region. It may be that the best one can do is to have a singularity at the wall itself which is still physically acceptable. This may also provide a mechanism for removing the zero-modes of the vacuum solution, since it is conceivable that varying the parameters χ_p would inevitably move one or more of the singularities in ϕ into the physical region.

Although the formalism extends to other simply-laced algebras, the application to these algebras is hampered by the relatively complicated technology for tau-functions other than those of the $a_n^{(1)}$ series. Whilst some partial results were obtained in these cases, it is clear that more powerful methods are required. Clearly one way forward is to try and adopt the methods of [23] which provide exact (if not explicit) formulae for multi-soliton solutions which are equally neat for any algebra. In their notation the tau-functions can be expressed as

$$\tau_j = \langle \Lambda_j | e^{-\frac{1}{2}E_1(t+x)} g e^{-\frac{1}{2}E_{-1}(t-x)} | \Lambda_j \rangle e^{-\frac{n_j}{4}(t^2-x^2)} \tag{6.1}$$

and the condition (2.16) can be written neatly as

$$\langle \Lambda_j | (Cn_j + E_1) g (Cn_j - E_{-1}) | \Lambda_j \rangle = 0 \; , \tag{6.2}$$

where g is an element of the group associated with the affine algebra. It would be interesting to see if this yields a more complete solution to the problem.

Finally, it is worth re-emphasising that analytically continued solitons seem to play an important role in real coupling Toda theory on the half-line and that this provides a respectable home for them. The model seems to be on a firmer footing than imaginary coupling theories with their manifest unitarity problems, and the singularities that inevitably occur for the solitons in the real-coupling theories can be placed behind the boundary out of harms way.

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